On the Complex Resonant Frequency of Open Dielectric Resonators

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Abstract—**An analytical method is presented for calculating accurately the complex resonant frequency of dielectric pillbox resonators. In this method, an approximated field of the resonator is expanded into a truncated series of solutions of the Helmholtz equation in the spherical coordinates, and the boundary condition on the resonator surface is treated in the least-squares sense. The resonant frequency and the intrinsic Q value due to radiation loss are obtained in the form of approximation converging to the exact values. Numerical results are compared with previously published calculations, which show that the present method is a relatively simple and effective one.**

I. Introduction

Dielectric pillbox resonators have found many practical applications, particularly in the spectral range from microwave to short millimeter-wave frequencies [1]–[7]. As far as the resonant frequency is concerned, the analyses for such a resonator have been performed by using several approximate methods, such as a magnetic wall model [7]–[10], a variational model [11], a dielectric waveguide model [12], [13], and a mixed model [14]. The method based on a magnetic wall model typically gives rise to numerical values smaller than experimental ones by about 10 percent, whereas the variational method of Konishi *et al.* [11] shows agreement with experimental results within 1 percent. On the other hand, Itoh *et al.* [12] have shown that the dielectric waveguide model agrees closely with Konishi's method, and also Garault *et al.* [14] have shown that a mixture of the magnetic wall and dielectric waveguide models can predict the resonant frequency to within less than 1-percent error. However, these approximate methods give no information about the radiation loss which always exists in dielectric resonators of the open type. It is expected that this kind of resonator will increase in use, particularly in the millimeter-wave region. Hence, accurate information about the intrinsic Q value due to radiation loss as well as that about the resonant frequency becomes important.

One effective approach to solve this problem has been presented by Van Bladel *et al.* [15]–[17]. They have analyzed an open dielectric resonator by the asymptotic expansion method in which the fields are expanded in powers of the reciprocal of refractive index $\sqrt{\epsilon_r}$. They performed calculations by considering only the first-order correction in $1/\sqrt{\epsilon_r}$.

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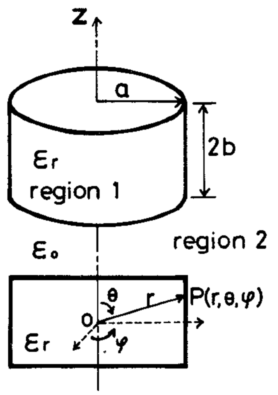

Fig. 1. Dielectric pillbox resonator and spherical coordinate system.

In this case, their method can avoid repeating calculations for every new value of ϵ_r. However, the validity of their method is limited to the case of high dielectric constants. In fact, they point out [17] that comparison with a dielectric sphere shows the accuracy to be, for the lowest mode and $\epsilon_r = 100$, of the order of 1 percent for the resonant frequency and 10 percent for the Q value. For $\epsilon_r = 35$, the accuracies worsened by a factor of the order of two. Although their method can derive more accurate results for the complex resonant frequency by consideration of high-order correction terms in $1/\sqrt{\epsilon_r}$, its treatment becomes very complicated.

In this paper, we present a new analytical method which analyzes accurately and easily the complex resonant frequency of an open dielectric pillbox resonator without a limit on the dielectric constant ϵ_r. This method expands the fields into a series of the solutions of the Helmholtz equation in spherical coordinates. In practice, however, such an infinite expansion requires truncation in a finite number of terms, and the boundary condition is treated in least-squares sense [18]. The analytical approach presented here is accurate in the sense that the resonant frequency and the intrinsic Q values due to radiation loss converge to the exact values as the number of terms in the truncated series is increased.

II. Analysis

Fig. 1 shows the geometry of an isolated dielectric pillbox resonator, for which the radius is a, the thickness is $2b$, and the relative dielectric constant is ϵ_r. The dielectric material is assumed to be isotropic and lossless throughout

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this paper. The major difficult in the analysis of this type of dielectric resonator lies in treating the irregular boundary of the structure which does not coincide with separable geometry. We expand the fields in region 1 and region 2 in terms of solutions to the Helmholtz equation in the spherical coordinate system (r, θ, φ) obtained by separation of variables. Using the scalar potentials ψ_{ri} and $\bar{\psi}_{ri}$ $(i=1,2)$, the fields of a resonator can be expressed as follows [19]:

$$\left.\begin{aligned} E_{ri} &= \left(\frac{\partial^2}{\partial r^2} + k_i^2\right)\psi_{ri} \\ E_{\theta i} &= \frac{1}{r}\frac{\partial^2 \psi_{ri}}{\partial r \partial \theta} - \frac{j\omega\mu_0}{r\sin\theta}\frac{\partial \bar{\psi}_{ri}}{\partial \varphi} \\ E_{\varphi i} &= \frac{1}{r\sin\theta}\frac{\partial^2 \psi_{ri}}{\partial r \partial \varphi} + \frac{j\omega\mu_0}{r}\frac{\partial \bar{\psi}_{ri}}{\partial \theta} \\ H_{ri} &= \left(\frac{\partial^2}{\partial r^2} + k_i^2\right)\bar{\psi}_{ri} \\ H_{\theta i} &= \frac{j\omega\epsilon_0\epsilon_{ri}}{r\sin\theta}\frac{\partial \psi_{ri}}{\partial \varphi} + \frac{1}{r}\frac{\partial^2 \bar{\psi}_{ri}}{\partial r \partial \theta} \\ H_{\varphi i} &= -\frac{j\omega\epsilon_0\epsilon_{ri}}{r}\frac{\partial \psi_{ri}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial^2 \bar{\psi}_{ri}}{\partial r \partial \varphi} \end{aligned}\right\} \quad (1)$$

where

$$\left.\begin{aligned} \psi_{ri} &= \cos(m\varphi + \varphi_0)\sum_n A_{ni}\sqrt{k_i r}\, F_{n+1/2}(k_i r) P_n^m(\cos\theta) e^{j\omega t} \\ \bar{\psi}_{ri} &= \sin(m\varphi + \varphi_0)\sum_n \bar{A}_{ni}\sqrt{k_i r}\, F_{n+1/2}(k_i r) P_n^m(\cos\theta) e^{j\omega t} \\ & \qquad (i=1,2). \end{aligned}\right\} \quad (2)$$

In these expressions, A_{ni} and $\bar{A}_{ni}$ are modal expansion coefficients to be determined, φ_0 is an arbitrary phase angle, and k_i is the wavenumber in the region i. $P_n^m(\cos\theta)$ is the first kind associated Legendre function of order n, m and $F_{n+1/2}(k_i r)$ is given by

$$F_{n+1/2}(k_i r) = \begin{cases} J_{n+1/2}(k_1 r), & \text{for region 1} \\ H_{n+1/2}^{(2)}(k_2 r), & \text{for region 2} \end{cases}. \quad (3)$$

Here, $J_{n+1/2}$ and $H_{n+1/2}^{(2)}$ are the first kind of Bessel function and the second kind of Hankel function of order $n + 1/2$, respectively.

The characteristic angular resonant frequency Ω (complex value) is determined by considering the boundary condition on the resonator surface, that is, $n \times (\mathbb{E}_1 - \mathbb{E}_2) = 0$ and $n \times (\mathbb{H}_1 - \mathbb{H}_2) = 0$ (n is the unit vector normal to the surface). However, as mentioned before, the infinite series in (2) should be truncated to a finite number of terms $n = N$ for practical calculations. Such approximated fields never satisfy the above type of boundary condition. We therefore fit the approximated fields to this boundary condition in the sense of least-squares [18]. For this purpose, we introduce the mean-square error E in the boundary

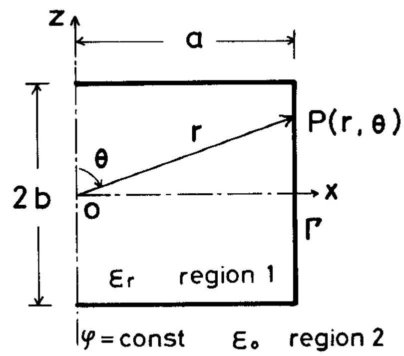


Fig. 2. Boundary contour Γ on the r-θ plane at an arbitrary φ coordinate.

condition, defined by the following equation:

$$\mathrm{E} = \int_s \left\{ |n \times (\mathbb{E}_1 - \mathbb{E}_2)|^2 + Z^2 |n \times (\mathbb{H}_1 - \mathbb{H}_2)|^2 \right\} ds \quad (4)$$

where Z, an arbitrary impedance parameter, is not uniquely defined, and the intrinsic impedance of the region 1, $Z_1 = \sqrt{\mu/\epsilon_0\epsilon_r}$, is used as Z in the following calculations. Here, the surface integral should be performed on the whole surface of resonator. However, if the geometry of the resonator has axial symmetry with respect to the Z axis, the above-mentioned surface integral can be reduced to the following line integral:

$$\mathrm{E} = \int_\Gamma \left\{ |\mathbb{E}_{t1} - \mathbb{E}_{t2}|^2 + Z_1^2 |\mathbb{H}_{t1} - \mathbb{H}_{t2}|^2 \right\} dl. \quad (5)$$

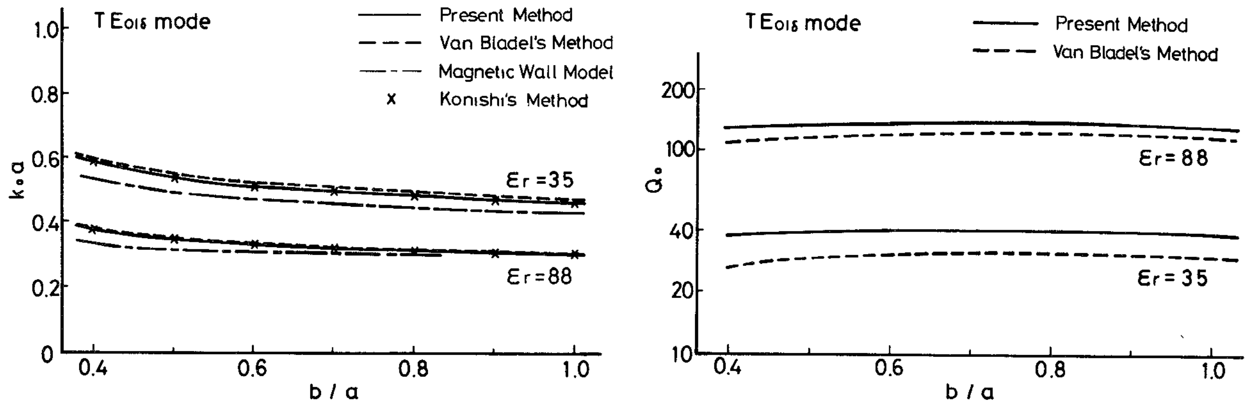
Here, Γ denotes the boundary contour on the $r - \theta$ plane at an arbitrary φ coordinate, as shown in Fig. 2, and $\mathbb{E}_{ti}, \mathbb{H}_{ti}$ $(i=1,2)$ denote the field components tangential to Γ. After substituting (1)–(3) into (5) and performing the integration numerically, we obtain error E as a function of both the modal coefficients and the angular frequency ω. The characteristic angular resonant frequency Ω is then obtained by means of the Ritz–Galerkin variational approach. We minimize E with respect to the above unknown variables, and obtain Ω by the same procedure as described in [18].

As a result, the characteristic angular resonant frequency for a mode is found as the complex quantity $\Omega = \Omega_r + j\Omega_i$, $(\Omega_r > 0, \Omega_i > 0)$, which leads to both the resonant frequency f_0 and the intrinsic Q value Q_0 due to radiation loss. These are given explicitly by

$$f_0 = |\Omega|/2\pi \quad Q_0 = |\Omega|/2\Omega_i. \quad (6)$$

III. Numerical Results

Since the pillbox resonator considered here has a plane of symmetry with respect to the $r - \varphi$ plane at $\theta = \pi/2$, symmetric and antisymmetric modes to this plane can exist independently. Then, the $r - \varphi$ plane at $\theta = \pi/2$ can be replaced with a magnetic wall without affecting the field distribution about which the E_φ-component is symmetric (or H_φ-component is antisymmetric). Similarly, if E_φ is antisymmetric (or H_φ is symmetric), an electric wall re-



(a) (b)

Fig. 3. Comparison of the numerical results for the $TE_{01\delta}$ mode between the present method and different methods. (a) Normalized resonant frequency. (b) Intrinsic Q value.

places that plane. Then, it is enough to consider only the boundary contour in the first quadrant for the integral contour of (5) if we utilize the following relation of the first kind associated Legendre function $P_n^m(\cos\theta)$ at $\theta = \pi/2$:

$$P_{m+q}^m(0) = \begin{cases} (-1)^{q/2}\cdot(q+2m-1)!!, & q = 0,2,4,\cdots \\ 0, & q = 1,3,5,\cdots \end{cases} \tag{7}$$

where

$$s!! = \begin{cases} 1, & s = -1,0 \\ s(s-2)\cdots 3\cdot 1, & s = 1,3,5,\cdots \\ s(s-2)\cdots 4\cdot 2, & s = 2,4,6,\cdots \end{cases}. \tag{8}$$

Now, for the case of $m = 0$, the field components expressed by (1) and (2) split into two independent groups of $(H_r, H_\theta, E_\varphi)$ and $(E_r, E_\theta, H_\varphi)$. The former group expressed by ψ_{rt} only become TE modes in the sense of $E_z = 0$, while the latter expressed by ψ_{ri} only become TM modes in the sense of $H_z = 0$. Such φ-independent modes are extensively used in most practical applications, and we hereafter investigate such modes only.

First, we compute the complex resonant frequency of the φ-independent $TE_{01\delta}$ model ($m = 0$). Table I shows the calculated results for different numbers N of expansion terms in (2) in order to investigate the convergence of both f_0 and Q_0. These calculations are performed for the structure with $\epsilon_r = 35$, 88, and $b/a = 1$. It is clear from this table that both the normalized resonant frequency k_0a and the intrinsic Q value Q_0 due to radiation loss almost converge for $N \geqslant 5$. Also, in Fig. 3(a) and (b), k_0a and Q_0 calculated for $N = 7$ are compared with those obtained by different methods. As seen in Fig. 3(a), the resonant frequency calculated by the present method agrees very well with those by Konishi's method [11], which gives agreement with experimental results to within 1 percent. Van Bladel's method, indicated by the dashed line, gives satisfactory results for the resonant frequency in case of $\epsilon_r = 88$, but its accuracy becomes worse for $\epsilon_r = 35$ because the magnitude

TABLE I
NORMALIZED RESONANT FREQUENCIES AND INTRINSIC Q VALUES CALCULATED FOR THE DIFFERENT NUMBER N OF THE EXPANSION TERMS

N	b / a = 1.0			
	εr = 35		εr = 88	
	k0a	Q0	k0a	Q0
1	0 474	0.443x10²	0.304	0 1 56x10³
2	0.473	0.433x10²	0.303	0 1 52x10³
3	0.470	0.401x10²	0.300	0.140x10³
4	0.469	0.400x10²	0.299	0.139x10³
5	0 467	0.395x10²	0 299	0.138x10³
6	0.467	0.395x10²	0.298	0.138x10³
7	0.467	0.393x10²	0.298	0.137x10³
8	0 467	0.393x10²	0.298	0.137x10³

of the correction term to the dominant one is proportional to $1/\sqrt{\epsilon_r}$. Q_0 is compared with those by Van Bladel's method in Fig. 3(b), and his results are found to be slightly lower than our results. This discrepancy may be caused by the same reason as mentioned above. In fact, this effect becomes more noticeable in Fig. 4(a) and (b) which shows k_0a and Q_0 as a function of the dielectric constant ϵ_r. The solid lines indicate the present results and the dashed lines indicate Van Bladel's results. It is obvious from these figures that his results approach our results with increasing ϵ_r, but the accuracy is poorer when ϵ_r is lower.

We have discussed so far the numerical results for the $TE_{01\delta}$ mode. We next investigate another φ-independent mode, i.e., the $TM_{01\delta}$ mode. Fig. 5(a) and (b) show k_0a and Q_0 calculated for $N = 7$, where the solid lines indicate the results by the present method and the cross marks indicate Van Bladel's results. Unlike the $TE_{01\delta}$ mode case, agreement between both methods is very good, even for lower ϵ_r. This feature can be understood from [15], since, when Van Bladel's method is followed for calculations of the $TM_{01\delta}$ mode, the first correction term in the asymptotic expansion is of order $1/\epsilon_r$, instead of order $1/\sqrt{\epsilon_r}$ in the case of the $TE_{01\delta}$ mode. Hence, his method gives more accurate results, especially for Q_0, for the $TM_{01\delta}$ mode than it does for the $TE_{01\delta}$ mode.

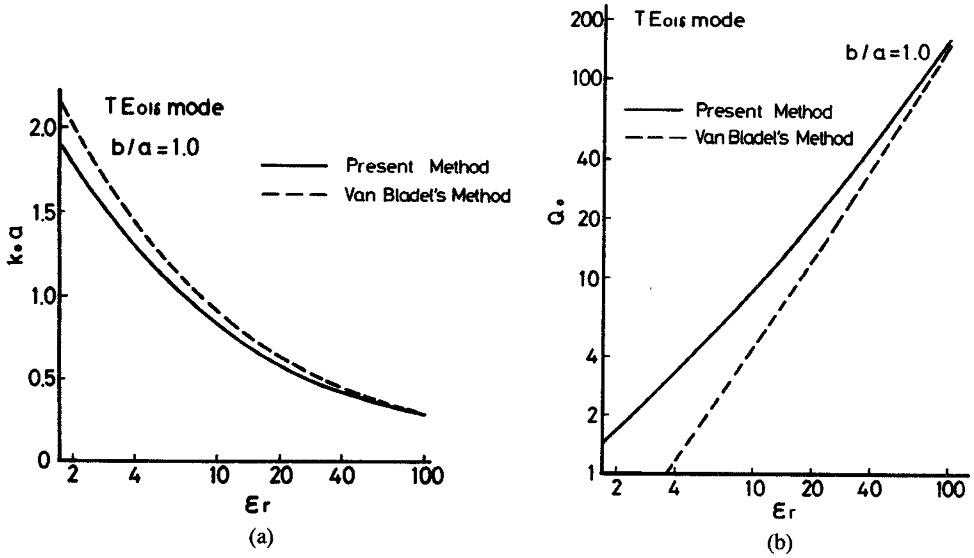


Fig. 4. Resonant characteristics of the $TE_{01\delta}$ mode as a function of the dielectric constant ϵ_r. (a) Normalized resonant frequency. (b) Intrinsic Q value.

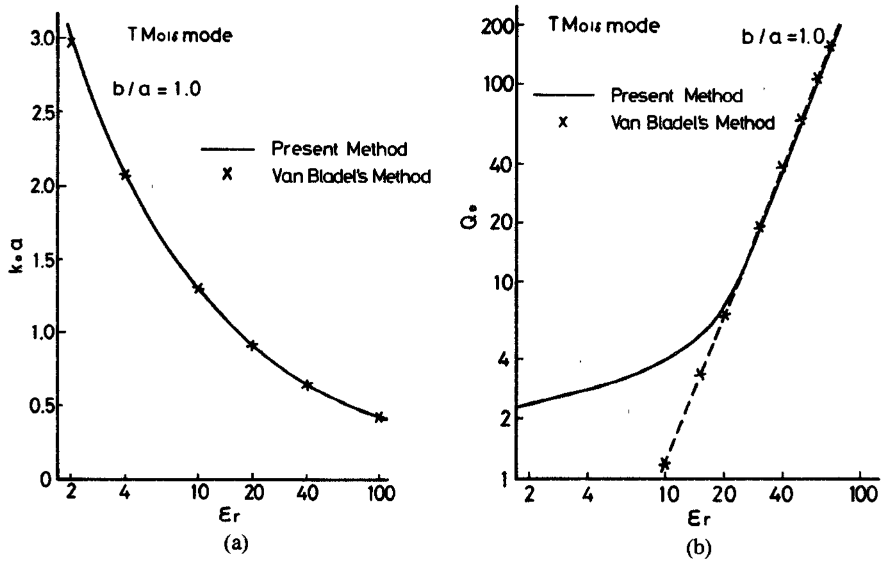


Fig. 5. Resonant characteristics of the $TM_{01\delta}$ mode as a function of the dielectric constant ϵ_r. (a) Normalized resonant frequency. (b) Intrinsic Q value.

Finally, in case of $m \neq 0$, the resonant modes become hybrid ones. For such modes, both ψ_{ri} and $\bar{\psi}_{ri}$ must be taken into account, so that the number of unknown coefficients to be determined becomes twice as much as that in case of $m = 0$. This point, however, does not cause any difficulty in calculations in the present method. A succeeding paper will present numerical discussions about important hybrid modes, as well as the experimental investigations.

IV. Conclusion

A new analytical method has been presented for calculating accurately the complex resonant frequency of an open dielectric pillbox resonator. The numerical discussion is presented for the $TE_{01\delta}$ and the $TM_{01\delta}$ modes, and the accuracy of the present method is confirmed with respect to both good convergence of calculations and comparison with previously published approximate methods. This method is based on the Rayleigh expansion theorem in which the approximate fields, expanded in the solution of the Helmholtz equation in the spherical coordinate system, satisfy the boundary conditions in the least-squares sense. Hence, the uniform convergence in the sequence of the truncated modal expansions such as in (2) can be assured mathematically [20]. In actual numerical calculations, however, one cannot always obtain precise solutions, in particular, for the problem with edge-shaped boundaries, though the method is complete in theory. This difficulty is due mainly to the slow convergence. Paying attention will be indeed necessary on this point even in the present problem, but special care is not taken into account; nevertheless a reasonable convergence is obtained in calculations as seen in Table I. Readers will find the detailed documents about the convergence and the analytical property of the Rayleigh expansions in [21] and [22].

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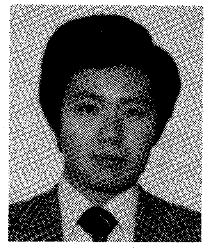

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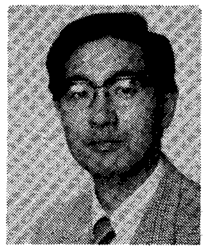

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